

Scale invariant cosmology II: model equations and properties

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ABSTRACT

Aims. We want to establish the basic properties of a scale invariant cosmology, that also accounts for the hypothesis of scale invariance of the empty space at large scales.

Methods. We write the basic analytical properties of the scale invariant cosmological models.

Results. The hypothesis of scale invariance of the empty space at large scale brings interesting simplifications in the scale invariant equations for cosmology. There is one new term, depending on the scale factor of the scale invariant cosmology, that opposes to gravity and favors an accelerated expansion. We first consider a zero-density model and find an accelerated expansion, going like $R(t) \sim t^2$. In models with matter present, the displacements due to the new term make a significant contribution Ω_λ to the energy-density of the Universe, satisfying an equation of the form $\Omega_m + \Omega_k + \Omega_\lambda = 1$.

Unlike the Friedman's models, there is a whole family of flat models ($k = 0$) with different density parameters $\Omega_m < 1$. We examine the basic relations between the density and geometrical properties, as well as the conservation laws. The models containing matter have an inflexion point, with first a braking phase followed by an accelerated expansion phase.

Conclusions. The scale invariant models have interesting properties and deserve further investigations

Key words. Cosmology: theory – Cosmology: dark energy – Cosmology: cosmological parameters

1. Introduction

The questions regarding the acceleration of the Universe expansion and the dark energy dominate the cosmological research for about two decades (Weinberg 1989; Carroll et al. 1992; Riess et al. 1998; Perlmutter et al. 1999; Frieman et al. 2008; Feng 2010; Porter et al. 2011; Solà 2013). The situation is like if an interaction of unknown nature opposes the gravitation at cosmological scales. A high number of different hypotheses have been formulated to try to explain the accelerated expansion.

There has been long-standing efforts to build a theory of gravitation, which also include scale invariance in addition to the invariance to a general coordinate transformation (Weyl 1923; Eddington 1923; Dirac 1973; Canuto et al. 1977). Scale covariant theories were often developed for trying to support the view that the gravitational constant G varies with time in relation with the so-called Large Number Hypothesis (Dirac 1973). Here, we do not follow this hypothesis and the gravitational constant remains a constant.

The laws of physics generally are not scale invariant, since the matter content of the medium considered may fix some scales of mass, length and time. However, the empty space as it is considered for example in the Minkowski metric has no preferred scale and in Paper I we have made the assumption that the empty space is scale invariant at large scales. This has lead to two differential relations between the cosmological constant and the scale factor λ , which expresses how the line element may change with time.

It is well known that at the quantum level, the properties of the vacuum are not scale invariant, since quantum physics defines units of length, time and mass. However, at the level of the Universe, especially in view of the problem of the accelerated expansion and dark energy, we really do not know whether

the assumption of scale invariance applies or not. This is what we want to explore on the basis of general field equations containing scale invariance in addition to the usual invariance to the group of transformations of curvilinear coordinates in Riemann space, which characterizes General Relativity. This enlarges the group of invariances sub-tending the theory of gravitation, with an invariance also present in electromagnetism. General Relativity appears as particular case of the new approach, when the scale factor is kept constant through space-time.

We make an explicit use of the two differential relations derived from the assumption of scale invariance of the empty space at large scales. These relations play an essential role and lead to solutions showing cosmic acceleration.

Sect. 2 gives the basic equations of scale invariant cosmology. In Sect. 3, we examine the case of an empty Universe in the scale invariant context. The critical density, the Ω and geometrical parameters are studied in Sect. 4. In Sect. 5, the appropriate conservation laws for a scale invariant cosmology are derived. Sect. 6 contains the conclusions.

2. The equations of scale invariant cosmology and properties

The scale or gauge invariant cosmology assumes that the field equations are invariant to a transformation of the line element like $ds' = \lambda(x^\mu) ds$, where $ds'^2 = g'_{\mu\nu} dx^\mu dx^\nu$ is the line element in General Relativity, while $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is the line element in a more general framework where scale invariance is supposed to also be a fundamental property. The quantities in the framework of General Relativity are noted with a prime, while those in the more general framework, that includes scale invariance, are without a prime. The parameter λ is the scale factor connecting

the two line elements. According to the Cosmological Principle of homogeneity and isotropy, λ can only depend on the cosmic time t .

The scale invariant field equation in cotensorial form has been given in Eq. (21) of Paper I. The general field equation has been applied to the empty space with the Minkowski metric. This has lead to two differential equations, which will play an essential role in the present work,

$$3 \frac{\dot{\lambda}^2}{\lambda^2} = \lambda^2 \Lambda_E, \quad \text{and} \quad 2 \frac{\ddot{\lambda}}{\lambda} - \frac{\dot{\lambda}^2}{\lambda^2} = \lambda^2 \Lambda_E. \quad (1)$$

They can also be written in equivalent forms,

$$\frac{\dot{\lambda}}{\lambda} = 2 \frac{\dot{\lambda}^2}{\lambda^2} \quad \text{and} \quad \frac{\ddot{\lambda}}{\lambda} - \frac{\dot{\lambda}^2}{\lambda^2} = \frac{\lambda^2 \Lambda_E}{3}. \quad (2)$$

The dots indicate the time derivatives and Λ_E is the Einstein cosmological constant, (we do not assign a prime to it, since there is here no ambiguity). These relations, derived from the hypothesis of the scale invariance of the empty space at large scales, express some interesting results:

- There is a relation of the cosmological constant Λ_E with the scale factor λ and its derivatives.
- There may be an energy-density associated to the time-variations of the scale factor.
- The first of equations (2) gives the time dependence of $\lambda(t)$,

$$\lambda = \sqrt{\frac{3}{\Lambda_E}} \frac{1}{c} t. \quad (3)$$

If we choose λ to be unity at the present time t_0 , then we have $\lambda = t_0/t$. As pointed out in Paper I, the first of equations (2) does not imply a particular origin for the time t . The origin will depend on the model considered. This also means that the amplitude of the variations of $\lambda(t)$ over the evolution of the Universe, from the origin to now, will strongly depend on the considered cosmological model.

The metric appropriate to cosmological models is the Robertson-Walker metric, characteristic of the homogeneous and isotropic space. A first step towards the equations we want to use can be derived in various equivalent ways (Canuto et al. 1977): – by expressing the general cotensorial field equation with the Robertson-Walker metric, – by taking advantage that there is a conformal transformation between the metrics $g'_{\mu\nu}$ and $g_{\mu\nu}$, – by applying a scale transformation to the current equations of cosmologies in $R(t)$, \dot{R} , \ddot{R} . The details of this third possibility for obtaining the basic equations are given in Appendix A. These equations are,

$$\frac{8\pi G \varrho}{3} = \frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + 2 \frac{\dot{R}}{R} \frac{\dot{\lambda}}{\lambda} + \frac{\dot{\lambda}^2}{\lambda^2} - \frac{\Lambda_E \lambda^2}{3} \quad (4)$$

and

$$-8\pi G p = \frac{k}{R^2} + 2 \frac{\ddot{R}}{R} + 2 \frac{\ddot{\lambda}}{\lambda} + \frac{\dot{R}^2}{R^2} + 4 \frac{\dot{R}}{R} \frac{\dot{\lambda}}{\lambda} - \frac{\dot{\lambda}^2}{\lambda^2} - \Lambda_E \lambda^2. \quad (5)$$

These equations contain several additional terms with respect to the standard case. In the same way as Λ_E , which is related to the energy-density of the vacuum, intervenes in the Λ CDM model, expressions (1) and (2) for the empty space, which characterize

λ and its properties also apply. Thus, with (1) and (2), the two above cosmological equations (4) and (5) may be simplified and become,

$$\frac{8\pi G \varrho}{3} = \frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + 2 \frac{\dot{R}}{R} \frac{\dot{\lambda}}{\lambda} \quad (6)$$

and

$$-8\pi G p = \frac{k}{R^2} + 2 \frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + 4 \frac{\dot{R}}{R} \frac{\dot{\lambda}}{\lambda}. \quad (7)$$

The combination of these two equations leads to

$$-\frac{4\pi G}{3} (3p + \varrho) = \frac{\ddot{R}}{R} + \frac{\dot{R}}{R} \frac{\dot{\lambda}}{\lambda}. \quad (8)$$

There, G is the gravitational constant, a real constant, k is the curvature parameter which takes values 0 or ± 1 , p and ϱ are the pressure and density in the scale invariant system of coordinates. Einstein cosmological constant has disappeared from the equations due to the account of the properties of the vacuum at macroscopic or large scales. Interestingly enough, these three equations differ from the classical ones, in each case only by the presence of one additional terms containing $\dot{R} \dot{\lambda}/(R \lambda)$. This additional term is different from zero if $\lambda(t)$ is not a constant. Indeed, if $\lambda(t)$ is a constant, one gets the usual equations of cosmologies for the expansion term $R(t)$. This means that at any fixed time, the effects which do not depend on the time evolution of the Universe, are just those predicted by General Relativity.

What is the significance of this additional term? Let us consider (8). The term on the left represents the attractive gravitational potential due to the matter and energy present in the considered model. This term contributes negatively to the second derivative of R and thus produces a braking of the motion of the comoving particles. The second term on the right of (8) is negative, since according to (3) we have

$$\dot{\lambda}/\lambda = -\frac{1}{t}. \quad (9)$$

This second term represents *an acceleration that opposes the gravitation*, it depends on both the Hubble constant \dot{R}/R and on the relative change $\dot{\lambda}/\lambda$ of the scale factor. This term may have a significant effect on the evolution of the Universe producing an acceleration, which may particularly reveal itself during the advanced stages of evolution, since according to (8) the acceleration is proportional to the relative velocity of the expansion. Equations (6) to (8) incorporate the scale invariance of the field equations as well the scale invariance properties of the vacuum at large scales. Numerical models have to be calculated to provide the solutions corresponding to the various choices of density and curvature parameters.

3. The case of an empty Universe

Let us first consider in the scale invariant framework the interesting case of a non-static model Universe with no matter nor significant radiation ($\varrho = 0$ and $p = 0$). The corresponding model in the standard case would be the empty Friedman model, (such a model with $k = -1$ has an expansion given by $R(t) \sim t$). We note that empty models are interesting as they represent the asymptotic limit of models with lower and lower densities. Moreover,

we may be confident that empty models should be scale invariant, since there is no matter present able to provide any scale. Expression (8) becomes simply

$$\frac{\ddot{R}}{R} = -\frac{\dot{R}\dot{\lambda}}{R\lambda}. \quad (10)$$

The integration with (9) gives $\dot{R} = at$, where a is a constant, a further integration gives

$$R = a(t^2 - t_{\text{in}}^2). \quad (11)$$

$R(t)$ grows like t^2 , the initial instant t_{in} of the model is chosen at the origin $R(t_{\text{in}}) = 0$ of the considered model, (t_{in} is not necessarily 0). The model is non-static and the Hubble value at time t is

$$H = \frac{\dot{R}}{R} = 2 \frac{t}{(t^2 - t_{\text{in}}^2)} \quad (12)$$

We do not know yet t_{in} , however we may use Eq. (6) and get,

$$\dot{R}^2 t - 2\dot{R}R + kt = 0, \quad (13)$$

This equation leads to a second expression of the Hubble constant,

$$H = \frac{\dot{R}}{R} = \frac{1}{t} \pm \frac{\sqrt{1 - k \frac{t^2}{R^2}}}{t}. \quad (14)$$

For an empty model, we may take $k = -1$ or $k = 0$ (this is consistent with (40) in the study of the geometrical parameters below). Let us first consider the case $k = -1$. The dimensions of k go like $[R^2/t^2]$. Fixing the scale so that $t_0 = 1$ and $R_0 = 1$ at the present time, we get from (14), the present Hubble constant H_0 being positive,

$$H_0 = \frac{1 + \sqrt{2}}{t_0}. \quad (15)$$

For $k = -1$, the above value represents a lowest bound of H_0 -values, expressed as a function of $t_0 = 1$, to the models with non-zero densities, (since the steepness of $R(t)$ increases with higher densities according to (6)). As is usual, the value of H_0 should be expressed in term of the age $\tau = t_0 - t_{\text{in}}$ of the Universe in the considered model. We find t_{in} by expressing the equality of the two values of H_0 obtained by (12) and (15),

$$\frac{t_{\text{in}}}{t_0} = \sqrt{2} - 1. \quad (16)$$

This is the minimum value of t_{in} , (we notice that here the scale factor $\lambda(t)$ at the origin has a value limited to $1 + \sqrt{2} = 2.4142$). The corresponding age τ becomes $\tau = (2 - \sqrt{2})t_0$ and we may now express the value of H_0 from (15) as a function of the age τ of the Universe. We have quite generally, indicating here in parenthesis the timescale referred to,

$$\frac{H_0(\tau)}{\tau} = \frac{H_0(t_0)}{t_0}, \quad \text{thus } H_0(\tau) = H(t_0)\tau. \quad (17)$$

Thus, we get the following value of $H_0(\tau)$, which is a maximum value, resulting from the fact that τ is a maximum,

$$H_0(\tau) = \sqrt{2} \quad \text{for } k = -1. \quad (18)$$

Let us now turn to the empty model with $k = 0$. We have according to (14)

$$H_0 = \frac{2}{t_0}, \quad (19)$$

As for $k = -1$, this value for the empty space is the minimum value of H_0 expressed in the scale with $t_0 = 1$. The comparison of (19) and (12) leads to $t_{\text{in}} = 0$ and thus $\tau = t_0$. Here also, t_{in} is the minimum value for all models with $k = 0$ and thus τ is the longest age. We have, expressing H_0 as a function of τ ,

$$H_0(\tau) = 2, \quad \text{for } k = 0. \quad (20)$$

Here also, $H_0(\tau)$ is an upper bound for the models with $k = 0$, due to the fact that the above τ is a maximum. We see that the empty models, whether $k = -1$ or $k = 0$, obey very simple properties.

The empty scale invariant model Universe expands like t^2 , thus with a strongly accelerating expansion over the ages. It expands much more rapidly than the corresponding Friedman model, which experiences a linear expansion $R \sim t$, with $H_0 = 1/t$, and shows no acceleration. Thus, here the effects of scale invariance appear as the source of a strongly accelerated expansion, consistently with the remark made above about relation (8).

4. Cosmological properties and parameters

We now examine some general properties and interesting parameters of the cosmological models based on equations (6) - (8).

4.1. Critical density and Ω -parameters

The critical density corresponding to the case $k = 0$ of the flat space is an essential model reference. Since the basic equations are different from the Friedman models, the critical density is also defined by a different expression. From (6) and (3), we have

$$\frac{8\pi G \varrho_c^*}{3} = H^2 - 2 \frac{H}{t}. \quad (21)$$

We mark with a * this critical density that does not correspond to the usual definition,

$$\varrho_c^* = \frac{3H^2}{8\pi G} \left(1 - \frac{2}{tH}\right). \quad (22)$$

Expression (22) evidently also applies for the critical density at present time t_0 , with a Hubble value H_0 . This critical density (22) is smaller than the corresponding critical density of Friedman models with $k = 0$. The parenthesis in (22) is always positive. This is true at any time t , since $2/(tH) = 2(dt/t)(R/dR)$ and the relative growth rate for non empty models is higher than t^2 . Indeed, we have seen that models satisfying relation (6) and with $k = 0$ (resp. $k = -1$) have a value of $H_0 \geq \frac{2}{t_0}$ (resp. $H_0 \geq \frac{1+\sqrt{2}}{t_0}$) according to (20) (resp. (18)). Thus, we have, for $k = 0$, $\frac{2}{t_0 H_0} \leq 1$

and, for $k = -1$, $\frac{2}{t_0 H_0} \leq \frac{2}{1+\sqrt{2}} = 0.828$, so that the parenthesis is zero or positive.

Let us now examine the various contributions to the mass and energy. Expressing (6) at time t and dividing by H^2 , we get with (3)

$$\frac{8\pi G \varrho}{3H^2} - \frac{k}{R^2 H^2} + \frac{2}{Ht} = 1. \quad (23)$$

If we now introduce the expression (22) for ϱ_c^* , we get

$$\frac{\varrho}{\varrho_c^*} - \frac{k}{R^2 H^2} + \frac{2}{Ht} \left(1 - \frac{\varrho}{\varrho_c^*}\right) = 1. \quad (24)$$

With the definitions,

$$\Omega_m^* = \frac{\varrho}{\varrho_c^*}, \quad \text{and} \quad \Omega_k = -\frac{k}{R^2 H^2}, \quad (25)$$

expression (24) becomes

$$\Omega_m^* + \Omega_k + \frac{2}{Ht} (1 - \Omega_m^*) = 1. \quad (26)$$

The quantity Ω_m^* is the ratio of the density to the critical density in the framework of the scale invariant theory. We see that $\Omega_m^* = 1$ implies $\Omega_k = 0$ and reciprocally, consistently with the definition of the critical density.

If, as in Sect. 3, we consider a vanishing density $\Omega_m^* \rightarrow 0$ for $k = -1$, this last equation tends to $0 + \Omega_k + \frac{2}{1+\sqrt{2}}(1 - 0) = 1$.

It implies $\Omega_k = 3 - 2\sqrt{2} = 0.1716$, thus according to (25) we get $H_0 = (1 + \sqrt{2})/t_0$ in agreement with the previous derivation (15). This is also consistent in the case $k = 0$, introducing (19) in (26) implies for $\Omega_k = 0$ for a zero density.

It will certainly be very useful for future comparisons with observational values to also consider the usual definition of the critical density, defined as in the framework of Friedman's models (this density is indicated without a *),

$$\Omega_m = \frac{\varrho}{\varrho_c} \quad \text{with} \quad \varrho_c = 3H^2/(8\pi G). \quad (27)$$

From the definition (22), the two density parameters are related by

$$\Omega_m = \Omega_m^* \left(1 - \frac{2}{Ht}\right). \quad (28)$$

The relation between these two Ω -parameters will be studied from numerical models. With Ω_m , relation (26) becomes simply,

$$\Omega_m + \Omega_k + \Omega_\lambda = 1, \quad (29)$$

$$\text{with} \quad \Omega_\lambda = \frac{2}{Ht}. \quad (30)$$

These two relations can also be derived directly by dividing (6) by H^2 and using (9). It also corresponds to the above relation (23). There, Ω_m is defined by (27), Ω_k by (25) and Ω_λ by (30).

The above relations evidently also apply at time t_0 , with the appropriate H_0 and t_0 . As mentioned above, Ω_λ must necessarily be smaller than 1 for models with $k = 0$ and than 0.828 for models with $k = -1$. It is a fortunate circumstance that an equation of the form of (29) is also valid in scale invariant cosmology. The difference with the standard case is that the term Ω_λ arising from scale invariance has replaced the usual term Ω_Λ due to the cosmological constant or dark energy. This term arises naturally from scale invariance and does not demand the existence of unknown particles.

We may also write relation (28) between the two density parameters as follows,

$$\Omega_m = \Omega_m^* (1 - \Omega_\lambda). \quad (31)$$

An equivalent and useful form is also

$$\Omega_m^* = \frac{\Omega_m}{\Omega_m + \Omega_k}. \quad (32)$$

These expressions allow us to make some further remarks on the Ω -parameters:

Case $k = 0$:

1. From (32), since $\Omega_k = 0$, we have $\Omega_m^* = 1$ and this applies at all times in a model.
2. The ratio $2/(tH)$ is not a constant (except for empty models, Sect. 3) and thus Ω_m is not a constant according to (28), it varies with age in a given model, see also Sect. (5.2) for the detailed behavior the Ω -parameters in the past. The balance between Ω_λ and Ω_m changes with time.
3. In Friedman's models, there is only one model corresponding to $k = 0$: the model with the critical density. In the scale invariant framework, for $k = 0$ the fact that $\Omega_m^* = 1$ does not imply specific values of Ω_m and Ω_λ . Thus, the additional term in (6) may lead to a variety of possible models for $k = 0$ with different parameters Ω_m and Ω_λ at time t_0 .
4. According to its definition (30) Ω_λ is positive, thus in order to satisfy (29) for $k = 0$, Ω_m must be smaller than 1. Thus, the variety of models for $k = 0$ consists in models with $\Omega_m < 1$.

Case $k = \pm 1$:

5. From (32), Ω_m^* is necessarily different from 1.
6. According to their definitions, the terms Ω_m , Ω_λ and Ω_k are expressed as functions of quantities, like $R(t)$, H , t , that change over the ages, thus these Ω -terms are not constant in time, (their behavior is examined in Sect. (5.2) on the basis of the conservation laws).
7. For $k = -1$, Ω_k is positive as well as Ω_λ , thus the variety of possible models must have $\Omega_m < 1$.
8. For $k = 1$, if $(\Omega_\lambda + \Omega_k) > 0$, there is only a variety of Ω_m -values smaller than 1. At this stage, we do not know the predicted range for the various Ω -parameters, but numerical results will confirm that the sum $(\Omega_\lambda + \Omega_k)$ is always positive for models with $k = 1$.

Depending on the values of Ω_m and Ω_k , the displacements associated to scale invariance could provide an important contribution Ω_λ to the energy-density present in the Universe. The CMB observations (de Bernardis et al. 2000), WMAP (Bennett et al. 2003) and the Planck Collaboration et al. (2015) support a flat model Universe with $k \approx 0$ with $\Omega_m \approx 0.30$ and $\Omega_\Lambda \approx 0.70$. We note that this last value is below the above permitted limit for Ω_λ . These values together with equation $\Omega_m + \Omega_k + \Omega_\lambda = 1$ would indicate that the energy associated to the effects resulting from scale invariance make a sizable fraction of the energy density of the Universe.

4.2. The geometry parameters

We now consider the geometry parameters k , $q_0 = -\frac{\ddot{R}_0 R_0}{\dot{R}_0^2}$ and their relations with Ω_m , Ω_k and Ω_λ at the present time t_0 . Expression (7) gives at the present time t_0 , if the pressure is zero,

$$\frac{k}{R_0^2} - 2q_0 H_0^2 + H_0^2 - 4\frac{H_0}{t_0} = 0. \quad (33)$$

Divided by H_0^2 and with (25), this becomes

$$-2q_0 + 1 - \Omega_k = \frac{4}{H_0 t_0}. \quad (34)$$

Eliminating Ω_k between (34) and (26), we obtain

$$2q_0 = \Omega_m^* - \frac{2}{H_0 t_0}(\Omega_m^* + 1), \quad (35)$$

and thus, if we use Ω_m rather than Ω_m^* ,

$$q_0 = \frac{\Omega_m}{2} - \frac{\Omega_\lambda}{2}. \quad (36)$$

This establishes relations between the acceleration parameter q_0 and the expressions of the matter content for a scale invariant cosmology. If $k = 0$ and thus $\Omega_m^* = 1$, we have from (35)

$$q_0 = \frac{1}{2} - \Omega_\lambda = \Omega_m - \frac{1}{2}, \quad (37)$$

which provides a very simple relation between basic parameters. For a present $\Omega_m = 0.30$, we get $q_0 = -0.20$. Such relations could also be considered at epochs different from the present one.

Most interestingly, the above basic relations are different from those of the Λ CDM. This is evidently expected since the basic equations (6) - (8) are different. Let us recall that in the Λ CDM model with $k = 0$ one has

$$q_0 = \frac{1}{2}\Omega_m - \Omega_\Lambda = \frac{3}{2}\Omega_m - 1 = \frac{1}{2} - \frac{3}{2}\Omega_\Lambda, \quad (38)$$

which may also be applied at different epochs. For $\Omega_m = 0.30$, we get $q_0 = -0.55$. In both cosmological, one has very simple relations expressing the q parameter. However, these expressions lead to significantly different results.

Let us now turn to the curvature parameter k . From the basic equation (6), we get

$$\frac{k}{R_0^2} = H_0^2 \left(\frac{8\pi G \varrho_0}{3H_0^2} - 1 + \frac{2}{t_0 H_0} \right). \quad (39)$$

With the definition of the critical density (22) and with (30), this becomes at the present time t_0 ,

$$\frac{k}{R_0^2} = H_0^2 \left[(\Omega_m^* - 1) \left(1 - \frac{2}{t_0 H_0} \right) \right], \quad (40)$$

which establishes a relation between k and Ω_m^* . It confirms that if $\Omega_m^* = 1$, one also has $k = 0$ and reciprocally. We also verify that for $2/(t_0 H_0) = 1$, we effectively have $k = 0$ in agreement with (19). Values of $\Omega_m^* > 1$ give a positive k -value, values smaller than 1 give a negative k -value. Using Ω_m , the above relation also writes at present time,

$$\frac{k}{R_0^2} = H_0^2 \left[\Omega_m - \left(1 - \frac{2}{t_0 H_0} \right) \right], \quad (41)$$

which is finally just equivalent to (29) at time t_0 .

We also have a relation between k and q_0 . From (35), we get

$$\Omega_m^* = \frac{2q_0 + \frac{2}{H_0 t_0}}{1 - \frac{2}{H_0 t_0}}, \quad (42)$$

and using this in (40), we obtain

$$\frac{k}{R_0^2} = H_0^2 \left[2q_0 - 1 + \frac{4}{H_0 t_0} \right]. \quad (43)$$

For $k = 0$, it evidently gives the same relation as from relation (36) above. We again emphasize that in all these expressions t_0 is not the present age of the Universe, but just the present time in a scale where $t_0 = 1$. As in Sect. 3, the present age $\tau = t_0 - t_{in}$, where the values of the initial time t_{in} depend on the considered model.

4.3. Inflexion point in the expansion

The Friedman models do not have an inflexion point, the second derivative \ddot{R} is always negative and thus q is positive at all times. In the scale invariant cosmology, like in the Λ CDM models, there are both a braking force of gravitational attraction and an acceleration force acting in the Universe model. There may thus be epochs dominated by gravitational braking and other epochs by acceleration. According to (8), there is an inflexion point in the curves $R(t)$, when we have the equality of braking and acceleration,

$$\frac{4\pi G}{3}(3p + \varrho) = \frac{H}{t}. \quad (44)$$

We better use (36), valid at any epoch t . For $q = 0$ in the scale invariant models, an inflexion in the curve $R(t)$ occurs at time t when

$$\Omega_m = \Omega_\lambda. \quad (45)$$

An inflexion point occurs when there is an equilibrium between these two Ω -parameters. The gravitational term dominates in the early epochs and the λ -acceleration dominates in more advanced stages. The higher the Ω_m -value, the later the inflexion point occurs. The empty model discussed in Sect. 3, where $R(t) \sim t^2$, seems to be an exception. It shows no inflexion point in the course of evolution and is starting with an horizontal tangent, before accelerating continuously. A non-zero density may lead to positive values of q at the origin followed by negative ones after the inflexion point. At this stage, we do not know whether scale invariant models predict an explosive origin.

For a flat model with $k = 0$, we can further precise the location of the inflexion point. Since in this case, $\Omega_m = 1 - \Omega_\lambda$, we have at the inflexion point

$$(1 - \Omega_\lambda) = \Omega_\lambda, \quad \text{and thus} \quad \Omega_\lambda = \Omega_m = \frac{1}{2}. \quad (46)$$

The matter and the λ -contributions should be equal and both equivalent to $1/2$. With (30), the inflexion point for $k = 0$ models occurs at time t such that

$$t = \frac{4}{H}, \quad (47)$$

where t is counted in the scale where $t_0 = 1$ at present and the same for H .

These results differ from those for the Λ CDM models. According to (38), we have $q = 0$ for a flat Λ CDM model when (Sutherland & Rothnie 2015),

$$\frac{1}{2} \Omega_m = \Omega_\Lambda. \quad (48)$$

This is to be compared to the scale invariant case given by (45). The acceleration term needs only to reach one half of the gravitational term to reach the critical limit in the Λ CDM model, while in the scale invariant case the inflexion point is reached for the equality of the two terms. This may provide possible observational tests, since the existence of an inflexion point in the evolution of the expansion factor $R(t)$ has been analyzed in several recent works (Melchiorri et al. 2007; Ishida et al. 2008; Sutherland & Rothnie 2015; Vitenti & Penna-Lima 2015; Moresco et al. 2016).

5. Conservation laws

5.1. General expression

The laws of conservation are fundamental properties of physics. It is clear that including a new invariance such as the scale invariance will influence in some way the laws of conservation. In addition, we have also explicitly accounted for the scale invariance of the vacuum at the macroscopic scales by using the differential equations (1) and (2). These various hypotheses have an impact on the conservation laws. We derive the conservation laws from the basic equations (6) to (8). We first rewrite (6) as follows and take its derivative,

$$8\pi G \varrho R^3 = 3kR + 3\dot{R}^2 R + 6\frac{\dot{\lambda}}{\lambda} \dot{R} R^2. \quad (49)$$

$$\begin{aligned} \frac{d}{dt}(8\pi G \varrho R^3) &= 3k\dot{R} + 3\dot{R}^3 + 6\dot{R}\ddot{R}R + \\ &+ 6\dot{R}R^2\frac{\dot{\lambda}}{\lambda} + 6\dot{R}R^2\frac{\ddot{\lambda}}{\lambda} + 12\dot{R}^2R\frac{\dot{\lambda}}{\lambda} - 6\dot{R}R^2\frac{\dot{\lambda}^2}{\lambda^2} \\ &= -3\dot{R}R^2 \left[-\frac{k}{R^2} - \frac{\dot{R}^2}{R^2} - 2\frac{\ddot{R}}{R} - 2\frac{\ddot{R}\dot{\lambda}}{\dot{R}\lambda} - 2\frac{\ddot{\lambda}}{\lambda} - 4\frac{\dot{R}\dot{\lambda}}{R\lambda} + 2\frac{\dot{\lambda}^2}{\lambda^2} \right]. \end{aligned} \quad (50)$$

We recognize terms belonging to the second member of (7), so that the above relation becomes

$$\frac{d}{dt}(8\pi G \varrho R^3) = -3\dot{R}R^2 \left[8\pi G p - 2\frac{\ddot{R}\dot{\lambda}}{\dot{R}\lambda} - 2\frac{\ddot{\lambda}}{\lambda} + 2\frac{\dot{\lambda}^2}{\lambda^2} \right]. \quad (51)$$

The scale invariance of the empty space imposes relations (2), which leads to further simplifications

$$\frac{d}{dt}(8\pi G \varrho R^3) = -3\dot{R}R^2 \left[8\pi G p - 2\frac{\ddot{R}\dot{\lambda}}{\dot{R}\lambda} - 2\frac{\dot{\lambda}^2}{\lambda^2} \right]. \quad (52)$$

Using (8) again, the third of our fundamental equations, to express the last two terms on the right of the above equation, we obtain

$$\frac{d}{dt}(8\pi G \varrho R^3) = -3\dot{R}R^2 \left[8\pi G p + \frac{R\dot{\lambda}}{\dot{R}\lambda} \left(8\pi G p + \frac{8\pi G \varrho}{3} \right) \right]. \quad (53)$$

We simplify by $8\pi G$, which is a constant, and write the above equation in differential form. With further simplifications it becomes,

$$3\lambda \varrho dR + \lambda R d\varrho + R \varrho d\lambda + 3p \lambda dR + 3p R d\lambda = 0. \quad (54)$$

$$\text{and} \quad 3\frac{dR}{R} + \frac{d\varrho}{\varrho} + \frac{d\lambda}{\lambda} + 3\frac{p}{\varrho} \frac{dR}{R} + 3\frac{p}{\varrho} \frac{d\lambda}{\lambda} = 0. \quad (55)$$

This can also be written in a form rather similar to the usual conservation law,

$$\frac{d(\varrho R^3)}{dR} + 3pR^2 + (\varrho + 3p)\frac{R^3}{\lambda} \frac{d\lambda}{dR} = 0. \quad (56)$$

These last two expressions are convenient forms of the law of conservation of mass-energy in the scale invariant cosmology. For a constant λ , we evidently recognize the conservation law or first integrals of the cosmological equations derived from General Relativity with the Robertson-Walker metric.

5.2. Specific cases: matter, radiation and vacuum

We now apply the above equation of conservation to some specific media characterized by different equations of state. We write the equation of state in the general form,

$$P = w\varrho, \quad (\text{with } c^2 = 1), \quad (57)$$

where w is taken here as a constant, (variable w depending on the epochs have been considered by some authors). The equation of conservation (55) becomes

$$3\frac{dR}{R} + \frac{d\varrho}{\varrho} + \frac{d\lambda}{\lambda} + 3w\frac{dR}{R} + 3w\frac{d\lambda}{\lambda} = 0, \quad (58)$$

with the following simple integral which covers all possible cases of constant w ,

$$\varrho R^{3(w+1)} \lambda^{(3w+1)} = \text{const.} \quad (59)$$

Different w -values correspond to different types of medium. For $w = 0$, we have the case of ordinary matter of density ϱ_m , exerting no pressure. We get

$$\varrho_m \lambda R^3 = \text{const.} \quad (60)$$

which means that the inertial and gravitational mass within a covolume should both (in agreement with the Equivalence Principle) slowly increase over the ages. At this stage, one may wonder how large are the changes of λ over the life of the Universe. For the empty model (Sect. 3), the change of λ is enormous, going from 1 at present to infinity at the origin. In a more realistic model, for example in a flat model with $\Omega_m = 0.30$, λ varies from 1 at present to about 1.4938 at the origin situated at $0.66943 t_0$ (cf. Paper III).

Although the effect of the variations of λ appears very limited in (60), how could we understand it? We do not expect any matter creation as in Dirac's Large Number Hypothesis (Dirac 1973) and thus the number of baryons should be a constant. However, as mentioned above, since an additional fundamental invariance has been accounted for, some changes in the conservation laws are necessarily to be expected. We note that a change of the inertial and gravitational mass is not a new fact, it is well known in Special Relativity, where the masses change as a function of their velocity. In the standard model of particle physics, the constant masses of elementary particles originate from the interaction of the Higgs field (Higgs 2014; Englert 2014) in the vacuum with originally massless particles. Here, the assumption of scale invariance of the vacuum (at large scales) and of the gravitation field would not let the mass invariant and make them to slowly slip over the ages, however by a limited amount in realistic models.

We do not know whether the present scale invariant models correspond to Nature. Some initial fundamental assumptions consistently lead to some consequences, however comparisons between models and observations may possibly confirm or infirm these results. This is why in a further paper we will proceed to model constructions and make such comparisons.

For now, we may check that the above expression (60) is fully consistent with the hypotheses made. From relation (17) derived from the study of the momentum-energy tensor in Paper I, we obtained that $\varrho' \lambda^2 = \varrho$, where we recall that the prime refers to the value in General Relativity and the symbols without a prime apply to the values in the scale invariant system. Thus expression (60) becomes, also accounting for the scale transformation $\lambda R = R'$,

$$\varrho_m \lambda R^3 = \varrho' \lambda^3 R^3 = \varrho' R'^3 = \text{const.} \quad (61)$$

This is just the usual mass conservation law in General Relativity.

Let us go on with the conservation law for relativistic particles and in particular for radiation with density ϱ_γ . Here, the ratio w of pressure to energy density is $w = 1/3$. From the equation of conservation (59), we get

$$\varrho_\gamma \lambda^2 R^4 = \text{const.} \quad (62)$$

There, a term λ^2 intervenes. As for the mass conservation, we may check its consistency with General Relativity. Expression (62) becomes $\varrho'_\gamma \lambda^4 R^4 = \text{const.}$ and thus $\varrho'_\gamma R'^4 = \text{const.}$ in the Einstein framework.

Another interesting case is that of the vacuum or dark energy (if any one) with density ϱ_v . It would obey to the equation of state $p = -\varrho$ with $c = 1$. Thus, we have $w = -1$ and (59) becomes

$$\varrho_v \lambda^{-2} = \text{const.} \quad (63)$$

suggesting a decrease of the vacuum energy over the ages. With $\varrho'_\gamma \lambda^2 = \varrho_v$, this corresponds to $\varrho'_\gamma = \text{const.}$ in the Einstein framework. This is the standard result, which corresponds to the presence of a cosmological constant in General Relativity.

We may now examine the time evolution of the Ω -parameters in a given model, in complement of the remarks in Sect. 4.1. In the matter dominated era, we have since $\varrho \sim t/R^3$ and $\varrho_c \sim H^2$,

$$\Omega_m \sim \frac{t}{R^3 H^2}. \quad (64)$$

For Ω_k and Ω_λ , the behaviors are like

$$\Omega_\lambda \sim \frac{1}{tH} \quad \text{and} \quad \Omega_k \sim \frac{1}{R^2 H^2}. \quad (65)$$

We remark that these three Ω -parameters would stay constant in time, only if the expansion factor $R(t)$ would go like t . This is the case neither for the empty model, nor for the models with some matter-density since they have both a braking and an acceleration phase. This confirms that the three Ω -parameters vary in time in scale invariant models, (evidently for $k = 0$ one has $\Omega_k = 0$ at all times). Let us now turn to the parameter Ω_m^* . We have seen that it is equal to 1 and remains constant in models with $k = 0$. What about the models with $k = \pm 1$? Let us examine the scaling predicted from (32),

$$\Omega_m^* = \frac{1}{1 + \frac{\Omega_k}{\Omega_m}} \sim \frac{1}{1 + \frac{R}{t}}. \quad (66)$$

There also R should go like t to maintain the constancy in time of Ω_m^* . As this is not the case, we conclude that Ω_m^* also varies with time in the models with $k = \pm 1$.

The above conservation laws are necessary for establishing the past matter and radiation history of the Universe, as well as for the integration of the cosmological equation (6). They will allow us to consider some terms as constant during the integration of the equations over the ages.

6. Conclusions

We have derived the equations of cosmologies in the scale invariant framework, also accounting for the scale invariance of the vacuum at large scales. This hypothesis brings interesting simplifications in the equations. On the whole, the scale invariant equations of cosmology only contain one additional term compared to the standard equations derived from General Relativity. The main physical consequence of this additional term is an acceleration of the cosmic expansion.

We first considered the model of a zero-density Universe. While in Friedman's models, the expansion of such a Universe model behaves like $R(t) \sim t$, in the scale invariant framework the model shows an accelerated expansion going like t^2 .

The main conclusion is that the contribution Ω_λ due to the effects of scale invariance to the energy density of the Universe is an important one, with $\Omega_m + \Omega_k + \Omega_\lambda = 1$. This energy density is in the form of the accelerated expansion. If this happens to apply, we might wonder about the need to invoke unknown particles.

For zero curvature $k = 0$, there is a whole family of models with different possible density parameters $\Omega_m < 1$. The geometrical parameters of the models and their relations with the

matter-density are also studied. The non-empty scale invariant models have an inflexion point with $q = 0$ in their evolution $R(t)$: there is first a gravitational braking of the expansion followed by a cosmic acceleration. The conditions for the inflexion point are not the same as for the Λ CDM models. The inclusion of the scale invariance modifies the conservation laws, which thus show a factor depending on the cosmic time.

On the whole, a consistent framework appears to exist for scale invariant cosmology. The observational tests will tell us whether this framework is worth to be further explored.

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Appendix A: Appendix: derivation of the basic equations

We derive the scale invariant equations in a straightforward way. Instead of applying the Robertson-Walker metric to the scale invariant field equation, we directly apply the scale transformations to the differential equations of cosmologies in the system of General Relativity. These equations are

$$\frac{8\pi G\rho'}{3} = \frac{k}{R'^2} + \frac{\dot{R}'^2}{R'^2} - \frac{\Lambda_E}{3}, \quad (\text{A.1})$$

$$-8\pi Gp' = \frac{k}{R'^2} + 2\frac{\ddot{R}'}{R'} + \frac{\dot{R}'^2}{R'^2} - \Lambda_E. \quad (\text{A.2})$$

There, Λ_E is the Einstein cosmological constant, G is the gravitational constant which is a real constant, k is the curvature parameter which may take values 0 and ± 1 , p' and ρ' are the pressure and density in the system of General Relativity coordinates. Now, we make the transformations

$$R' = \lambda R \quad \text{and} \quad dt' = \lambda dt. \quad (\text{A.3})$$

We get

$$\dot{R}' = \frac{dR'}{dt'} = \frac{\lambda \dot{R} + \lambda \dot{R}}{\lambda}, \quad (\text{A.4})$$

where the dot over a symbol indicates its derivative with respect to the time "t" in the scale invariant system. Then, we have

$$\frac{\dot{R}'}{R'} = \frac{1}{\lambda} \left(\frac{\dot{\lambda}}{\lambda} + \frac{\dot{R}}{R} \right). \quad (\text{A.5})$$

The second derivative \ddot{R}' becomes

$$\ddot{R}' = \frac{d(\frac{\dot{R}'}{dt'})}{dt'} = \frac{1}{\lambda^2} (\ddot{\lambda}R + 2\dot{\lambda}\dot{R} + \lambda\ddot{R}) - \frac{(\dot{\lambda}R + \lambda\dot{R})}{\lambda^2} \frac{\dot{\lambda}}{\lambda}, \quad (\text{A.6})$$

and

$$\frac{\ddot{R}'}{R'} = \frac{1}{\lambda^2} \left(\frac{\ddot{\lambda}}{\lambda} + \frac{\dot{\lambda}\dot{R}}{\lambda R} + \frac{\ddot{R}}{R} - \frac{\dot{\lambda}^2}{\lambda^2} \right). \quad (\text{A.7})$$

Thus, by replacing in (A.1) we obtain

$$\frac{8\pi G\rho}{3} = \frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + 2\frac{\dot{\lambda}\dot{R}}{\lambda R} + \frac{\dot{\lambda}^2}{\lambda^2} - \frac{\Lambda_E\lambda^2}{3} \quad (\text{A.8})$$

and from (A.2) after simplifications,

$$-8\pi Gp = \frac{k}{R^2} + 2\frac{\ddot{R}}{R} + 2\frac{\ddot{\lambda}}{\lambda} + \frac{\dot{R}^2}{R^2} + 4\frac{\dot{R}\dot{\lambda}}{R\lambda} - \frac{\dot{\lambda}^2}{\lambda^2} - \Lambda_E\lambda^2. \quad (\text{A.9})$$

The various quantities in the equations are expressed in the general system where scale invariance is a property. There, we have used the relations (17) of Paper I imposed by the scale invariance of the energy-momentum tensor, $p = p'\lambda^2$ and $\rho = \rho'\lambda^2$. These two relations correspond to the results by Canuto et al. (1977). At this stage, these relations do not account for the relations expressing the scale invariance of the empty space, which lead to substantial simplifications.

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